

FORMS OF HIGHER DEGREE PERMITTING COMPOSITION

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ABSTRACT. Nondegenerate forms N of degree d on a unital nonassociative algebra A over a ring R which permit composition, i.e., satisfy $N(1) = 1$ and $N(xy) = N(x)N(y)$ for all x, y in A , are studied. These forms were first classified by Schafer over fields of characteristic 0 or $> d$. We investigate cubic and quartic nondegenerate forms which permit composition over certain rings and curves. Classes of highly degenerate cubic forms N over fields which permit composition are constructed.

1. INTRODUCTION

Finite-dimensional forms of degree d permitting composition which are defined over fields of characteristic 0 or $> d$ were classified by Schafer (see [S1] for an overview, [S2] for cubic forms and [S3] for forms of higher degree). Both the restriction on the dimension of the underlying vector space to be finite, and on the characteristic of the base field which were needed in Schafer's proofs, were later omitted, respectively relaxed, by McCrimmon [M1]. Forms permitting composition over arbitrary base rings instead of fields were first studied by Baumgartner and Bergmann [B-B] in 1974, McCrimmon [M2] in 1985 and Petersson [P] in 1993. McCrimmon studied quadratic forms permitting composition in the context of nonassociative algebras with a scalar involution. Petersson generalized the classical Cayley-Dickson doubling process, and completely classified composition algebras over arbitrary rings which, as modules, are finitely generated projective with full support. Baumgartner and Bergmann investigated nondegenerate cubic forms over arbitrary commutative unital rings satisfying a certain multiplicativity condition which is a canonical generalization of what we call "permitting composition". In particular, any unital nonassociative algebra over R carrying a nondegenerate multiplicative cubic form mapping the unit element of the algebra onto the unit of the base ring, was shown to be alternative and algebraic [B-B, Satz 3, Satz 4]. This result was extended to arbitrary multiplicative forms of higher degree over rings R with $[d/2]! \in R^\times$ by Legrand-Legrand [L-L]. Since there are no restrictions on the characteristic of the base ring, the more general concept of a form of degree d as it was developed by Bergmann [B] was used (see also Roby [R]; the concept of forms developed by him comprises the one of Bergmann, both coincide for finitely generated projective modules). In the setting we consider, their forms are identical with ours and some of Schafer's results carry over verbatim to the setting of forms permitting composition over rings.

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Petersson was the first to investigate composition algebras over locally ringed spaces, thus initiating the study of nonassociative algebras over algebraic varieties. In particular, he generalized the classical Cayley-Dickson doubling process due to Albert [A] to this more general setting [P, 2.5]. Composition algebras (defined on locally free modules of constant finite rank) were classified over curves of genus zero in [P, 4.4].

We study forms of degree $d \geq 3$ permitting composition which are defined over rings with $d! \in R^\times$ instead of fields. We also investigate unital nonassociative \mathcal{O}_X -algebras \mathcal{A} over locally ringed spaces X , $d! \in H^0(X, \mathcal{O}_X^\times)$, which carry a nondegenerate form $N : \mathcal{A} \rightarrow \mathcal{O}_X$ of degree $d \geq 3$ permitting composition, i.e., $N(xy) = N(x)N(y)$ for all sections x, y of \mathcal{A} over the same open subset of X and $N(1_{\mathcal{A}}) = 1$.

The contents of this paper are as follows. Let R be a commutative ring such that $d! \in R^\times$. Let A be a unital nonassociative R -algebra which is finitely generated projective and faithful as an R -module. Suppose that A carries a nondegenerate form N of degree d permitting composition. Notation and basic facts are given in Section 1. After some straightforward generalizations of results in [S3] to forms over R in Section 2, the cases $d = 3$ and $d = 4$ are considered separately in Section 4. In particular, we investigate the properties of A over a domain. Examples of cubic and quartic forms over the rings $k[t]$ and $k[t, \frac{1}{t}]$ which permit composition are given in Section 5. In Section 6 our previous results are generalized to the setting of locally ringed spaces. In Section 7, algebras over a curve of genus zero which admit a nondegenerate cubic form N permitting composition are listed. As an application, which shows how useful the theory of alternative algebras over varieties can be, we construct classes of unital alternative algebras of degree 3 over a field of characteristic not 2 or 3 with a highly degenerate form N which permits composition in Section 8.

In the following, the standard terminology from algebraic geometry, see Hartshorne's book [H], is used.

2. PRELIMINARIES

2.1. Let R be a unital commutative associative ring. The *rank* of a finitely generated projective R -module M is defined as $\sup\{\text{rank}_{R_P} M_P \mid P \in \text{Spec } R\}$. Let A be a nonassociative unital R -algebra. The *nucleus* $\text{Nuc}(A) = \{x \in A \mid [x, A, A] = [A, x, A] = [A, A, x] = 0\}$ of A is the set of those elements in A , which associate with all elements in A . The *center* of A is the set of all elements which commute and associate with all elements of A ; that is, the set $Z(A) = \{x \in \text{Nuc}(A) \mid xy = yx \text{ for all } y \in A\}$ [S4, p. 14].

Remark 1. (i) An R -module M has *full support* if $\text{Supp}(M) = \{P \in \text{Spec } R \mid M_P \neq 0\} = \text{Spec } R$. Every nonassociative unital algebra over R which is finitely generated projective with full support is *faithful*; i.e., $rA = 0$ implies $r = 0$.

(ii) If A is finitely generated projective and faithful as R -module, $R1_A$ is a direct summand of A : The map $\epsilon : R \rightarrow A$, $\epsilon(r) = r1_A$ has a section if and only if $\text{Hom}(\epsilon, 1_A)$ is surjective. Since A_P has positive dimension, the map is surjective modulo any maximal ideal, which implies the assertion (see the proof of [Kn1, I.(1.3.5)]).

Unless stated otherwise, the term “ R -algebra” refers to unital nonassociative algebras which are finitely generated projective and faithful as R -modules.

2.2. An R -algebra C is called a *composition algebra* if it admits a quadratic form $N : C \rightarrow R$ such that $N(1_C) = 1$ which permits composition and whose induced symmetric bilinear form is *nondegenerate*; i.e., it determines an R -module isomorphism $C \xrightarrow{\sim} \check{C} = \text{Hom}_R(C, R)$. C is alternative and quadratic. The quadratic form N is uniquely determined by C and called the *norm* of C . Composition algebras over R only exist in ranks 1, 2, 4 or 8. Those of constant rank 2 are called *quadratic étale R -algebras*. Those of constant rank 4 are called *quaternion algebras* and the ones of constant rank 8 are called *octonion algebras*. If there is a nondegenerate quadratic form $N : A \rightarrow R$ on a unital nonassociative R -algebra A permitting composition, then A is a composition algebra over R and of rank 1, 2, 4 or 8 [P].

2.3. Let d be a positive integer and assume $d! \in R^\times$. Let M be a finitely generated projective R -module. A *form of degree d* (on M) over R is a map $N : M \rightarrow R$ such that $N(ax) = a^d N(x)$ for all $a \in R$, $x \in M$, and where the map $\theta : M \times \cdots \times M \rightarrow R$ defined by

$$\theta(x_1, \dots, x_d) = \frac{1}{d!} \sum_{1 \leq i_1 < \cdots < i_d \leq d} (-1)^{d-l} N(x_{i_1} + \cdots + x_{i_l})$$

is a d -linear form over R (the range of summation of l being $1 \leq l \leq d$). θ is called the *symmetric d -linear form* associated with N and (M, θ) a d -linear space. Obviously, $N(x) = \theta(x, \dots, x)$. Note that a symmetric R -module homomorphism $M \otimes \cdots \otimes M \rightarrow R$ or an R -module homomorphism $S_d(M) \rightarrow R$, where $S_d(M)$ is the symmetric algebra of M , also define a symmetric d -linear form.

N (or, respectively, the associated d -linear form θ) is called *nondegenerate*, if the map $M \rightarrow \text{Hom}_R(M \otimes \cdots \otimes M, R)$ ($(d-1)$ -copies of M)

$$\theta_{x_1}(x_2 \otimes \cdots \otimes x_d) = \theta(x_1, x_2, \dots, x_d)$$

is injective (cf. Keet [K, p. 4946]). This concept of nondegeneracy is not invariant under base change. A stronger notion, invariant under base change, is to require that $\theta_{x_1} \otimes k(P)$ is injective for all $P \in \text{Spec } R$. This is equivalent to saying that θ_{x_1} is an isomorphism of M onto a direct summand of $\text{Hom}_R(M^{\otimes(d-1)}, R)$. Both notions, however, are equivalent for forms permitting composition, see Lemma 1 (i).

Two d -linear spaces (M_i, θ_i) , $i = 1, 2$ are called *isomorphic* (written $(M_1, \theta_1) \cong (M_2, \theta_2)$ or just $\theta_1 \cong \theta_2$) if there exists an R -module isomorphism $f : M_1 \rightarrow M_2$ such that $\theta_2(f(x_1), \dots, f(x_d)) = \theta_1(x_1, \dots, x_d)$ for all $x_1, \dots, x_d \in M_1$.

The *orthogonal sum* $(M_1, \theta_1) \perp (M_2, \theta_2)$ of (M_i, θ_i) , $i = 1, 2$, is defined to be the R -module $M_1 \oplus M_2$ together with the d -linear form $(\theta_1 \perp \theta_2)(u_1 + x_1, \dots, u_d + x_d) = \theta_1(u_1, \dots, u_d) + \theta_2(x_1, \dots, x_d)$. A d -linear space (M, θ) is called *decomposable*, if $(M, \theta) \cong (M_1, \theta_1) \perp (M_2, \theta_2)$ for two non-zero d -linear spaces (M_i, θ_i) , $i = 1, 2$. A non-zero d -linear space (M, θ) is called *indecomposable* if it is not decomposable. We distinguish between indecomposable ones and *absolutely indecomposable* ones; i.e., d -linear spaces which stay indecomposable under base change.

2.4. Let X be a locally ringed space with structure sheaf \mathcal{O}_X . For $P \in X$ let $\mathcal{O}_{P,X}$ be the local ring of \mathcal{O}_X at P and m_P the maximal ideal of $\mathcal{O}_{P,X}$. The corresponding residue class field is denoted by $k(P) = \mathcal{O}_{P,X}/m_P$. For an \mathcal{O}_X -module \mathcal{F} the stalk of \mathcal{F} at P

is denoted by \mathcal{F}_P . \mathcal{F} is said to have *full support* if $\text{Supp } \mathcal{F} = X$; i.e., if $\mathcal{F}_P \neq 0$ for all $P \in X$. We call \mathcal{F} *locally free of finite rank* if for each $P \in X$ there is an open neighborhood $U \subset X$ of P such that $\mathcal{F}|_U = \mathcal{O}_U^r$ for some integer $r \geq 0$. The *rank* of \mathcal{F} is defined to be $\sup\{\text{rank}_{\mathcal{O}_{P,X}} \mathcal{F}_P \mid P \in X\}$. The term “ \mathcal{O}_X -algebra” (or “algebra over X ”) always refers to nonassociative \mathcal{O}_X -algebras which are unital and locally free of finite rank as \mathcal{O}_X -modules.

2.5. Let \mathcal{C} be an \mathcal{O}_X -algebra. \mathcal{C} is called a *composition algebra* over X if it has full support and if there exists a *nondegenerate* quadratic form $N: \mathcal{C} \rightarrow \mathcal{O}_X$ (i.e., the induced symmetric bilinear form $N(u, v) = N(u + v) - N(u) - N(v)$ determines a module isomorphism $\mathcal{C} \xrightarrow{\sim} \check{\mathcal{C}} = \mathcal{H}om(\mathcal{C}, \mathcal{O}_X)$), such that $N(uv) = N(u)N(v)$ for all sections u, v of \mathcal{C} over the same open subset of X [P, 1.6].

The form N is uniquely determined by these conditions and called the *norm* of \mathcal{C} . It is denoted by $N_{\mathcal{C}}$. Given an algebra \mathcal{C} over X and a quadratic form $N: \mathcal{C} \rightarrow \mathcal{O}_X$, the algebra \mathcal{C} is a composition algebra over X with norm N if and only if \mathcal{C}_P is a composition algebra over $\mathcal{O}_{P,X}$ with norm N_P for all $P \in X$. Composition algebras over X are invariant under base change, and exist only in ranks 1, 2, 4 or 8. A composition algebra of constant rank 2 (resp. 4 or 8) is called a *quadratic étale algebra* (resp. *quaternion algebra* or an *octonion algebra*). A composition algebra over X of constant rank is called *split*, if it contains a composition subalgebra isomorphic to $\mathcal{O}_X \oplus \mathcal{O}_X$ [P, 1.7, 1.8].

If X is an R -scheme with structure morphism $\tau: X \rightarrow \text{Spec } R$, then a composition algebra \mathcal{C} over X is *defined over R* if there exists a composition algebra C over R such that $\mathcal{C} = \tau^* C \cong C \otimes \mathcal{O}_X$. There exists a generalized Cayley-Dickson doubling $\text{Cay}(\mathcal{D}, \mathcal{P}, N)$ for a composition algebra \mathcal{D} of constant rank ≤ 4 over a locally ringed space [P, 2.3, 2.4, 2.5].

2.6. Let $d! \in H^0(X, \mathcal{O}_X^\times)$. Let \mathcal{M} be an \mathcal{O}_X -module which is locally free of finite rank. A *form of degree d* (on \mathcal{M}) over \mathcal{O}_X is a map $N: \mathcal{M} \rightarrow \mathcal{O}_X$ such that $N(ax) = a^d N(x)$ for all sections a of \mathcal{O}_X , x of \mathcal{M} over the same open subset of X , and where the map $\theta: \mathcal{M} \times \cdots \times \mathcal{M} \rightarrow \mathcal{O}_X$ defined by

$$\theta(x_1, \dots, x_d) = \frac{1}{d!} \sum_{1 \leq i_1 < \cdots < i_l \leq d} (-1)^{d-l} N(x_{i_1} + \cdots + x_{i_l})$$

($1 \leq l \leq d$) is a d -linear form over \mathcal{O}_X . θ is called the *symmetric d -linear form* associated with N . (This definition of associated d -linear form deviates from the one used for the associated symmetric bilinear form in 2.5 or [P] by the factor $1/2$ which was omitted in order to be able to include the case that $2 \notin H^0(X, \mathcal{O}_X^\times)$ in the classification result for composition algebras). A form N of degree $d \geq 3$ (or, respectively, its associated d -linear form θ) is called *nondegenerate*, if the map $\mathcal{M} \rightarrow \mathcal{H}om_X(\mathcal{M} \otimes \cdots \otimes \mathcal{M}, \mathcal{O}_X)$ ($(d-1)$ -copies of \mathcal{M}),

$$\theta_{x_1}(x_2 \otimes \cdots \otimes x_d) = \theta(x_1, x_2, \dots, x_d)$$

is injective.

Two d -linear spaces $(\mathcal{M}_i, \theta_i)$ ($i = 1, 2$) are called *isomorphic* if there exists an \mathcal{O}_X -module isomorphism $f: \mathcal{M}_1 \rightarrow \mathcal{M}_2$ such that $\theta_2(f(v_1), \dots, f(v_d)) = \theta_1(v_1, \dots, v_d)$ for all sections v_1, \dots, v_d of \mathcal{M}_1 over the same open subset of X .

The *orthogonal sum* $(\mathcal{M}_1, \theta_1) \perp (\mathcal{M}_2, \theta_2)$ of $(\mathcal{M}_i, \theta_i)$, $i = 1, 2$, is defined to be the \mathcal{O}_X -module $\mathcal{M}_1 \oplus \mathcal{M}_2$ together with the d -linear form $(\theta_1 \perp \theta_2)(u_1 + v_1, \dots, u_d + v_d) = \theta_1(u_1, \dots, u_d) + \theta_2(v_1, \dots, v_d)$. A d -linear space (\mathcal{M}, θ) is called *decomposable*, if $(\mathcal{M}, \theta) \cong (\mathcal{M}_1, \theta_1) \perp (\mathcal{M}_2, \theta_2)$ for two non-zero d -linear spaces $(\mathcal{M}_i, \theta_i)$, $i = 1, 2$. A non-zero d -linear space (\mathcal{M}, θ) is called *indecomposable* if it is not decomposable. We will distinguish between indecomposable ones and *absolutely indecomposable* ones; i.e., d -linear spaces which stay indecomposable under base change.

3. FORMS PERMITTING COMPOSITION

Large parts of the results and proofs of [S3] can be generalized verbatim to the case where the base field is replaced by an arbitrary commutative ring R with $d! \in R^\times$. We briefly summarize them for the sake of the reader.

Let R be a ring with $d! \in R^\times$. Unless explicitly stated otherwise, the term “ R -algebra” refers to unital nonassociative algebras which are finitely generated projective and faithful as R -modules. Let A be an algebra over R , $1 = 1_A$ the unit element of A and $N : A \rightarrow R$ a form of degree d permitting composition; i.e.,

$$\theta(xy, \dots, xy) = \theta(x, \dots, x)\theta(y, \dots, y)$$

for all $x, y \in A$ and $N(1_A) = 1$. Linearizing this equation in x and then in y we obtain

$$\theta(x_1 y, \dots, x_d y) = \theta(x_1, \dots, x_d)N(y),$$

and

$$\sum_{\sigma} \theta(x_1 y_{\sigma(1)}, \dots, x_d y_{\sigma(d)}) = d! \theta(x_1, \dots, x_d) \theta(y_1, \dots, y_d),$$

where σ ranges over all the permutations in S_d . This implies

$$\theta(xy_1, \dots, xy_d) = N(x)\theta(y_1, \dots, y_d)$$

by symmetry [S3, (5), (6) and (7)]. (Indeed, this observation does not require A to have a unit element.) For $i = 1, \dots, d$ we define a form $T_i : A \rightarrow R$ of degree i via

$$T_i(x) = \binom{d}{i} \theta(x, \dots, x, 1, \dots, 1) \quad (i\text{-times } x).$$

Then

$$N(x) = T_d(x) \text{ and } T_1(x) = d\theta(x, 1, \dots, 1).$$

Put $T_0(x) = 1$ and $T_{d+q}(x) = 0$ for $q > 0$. The form $T : A \rightarrow R$, $T(x) = T_1(x)$ is called the *trace*. Define $A_0 = \ker T$. A can be written as the direct sum of R -modules

$$A = R1_A \oplus A_0.$$

From now let $d \geq 3$, unless explicitly stated otherwise.

A d -linear form θ is *invariant under all left and right multiplications with elements of trace zero*, if

$$\theta(x_1 a, \dots, x_d) + \theta(x_1, x_2 a, \dots, x_d) + \dots + \theta(x_1, \dots, x_d a) = 0$$

and

$$\theta(ax_1, \dots, x_d) + \theta(x_1, ax_2, \dots, x_d) + \dots + \theta(x_1, \dots, ax_d) = 0$$

for all $x_i \in A$ and for all elements $a \in A$ of trace zero.

$B(x, y) = T(xy)$ is a symmetric bilinear form on A which is associative, and if N is nondegenerate, then so is B . A is alternative; i.e., $x^2y = x(xy)$ and $yx^2 = (yx)x$ for all elements $x, y \in A$. Every element $x \in A$ satisfies

$$x^d - T_1(x)x^{d-1} + T_2(x)x^{d-2} - \cdots + (-1)^dT_d(x)1 = 0$$

([S3, Theorem 2, Theorem 3], for $d = 2$ this was shown in [M2, 4.6]).

Remark 2. (i) If N is degenerate, the algebra A need not be alternative; for a counterexample for $d = 2$ see [M2, 4.13].

(ii) Baumgartner and Bergmann studied cubic forms N on unital nonassociative algebras A over arbitrary rings which they called multiplicative, and called such a pair (A, N) a *composition algebra of third degree*. Each nondegenerate composition algebra (in the sense of [B-B]) of degree 3 over an arbitrary ring is alternative [B-B, Satz 3]. This result for composition algebras of third degree in the sense of [B-B] was later generalized by Legrand-Legrand [L-L] to multiplicative forms of higher degree over rings with $[d/2]!$ invertible. Their definition of forms is different from ours in order to accommodate the case that $d!$ is not invertible in R . Over fields of characteristic not 2 or 3, or over the rings considered here, the different concepts of forms coincide and a form N permitting composition which satisfies $N(1_A) = 1$ corresponds to a multiplicative form. For the above equation in a more general context, see [L-L, Corollaire 2.2].

Theorem 1. *Suppose that R is a domain. Let A be an R -algebra, $\theta : A \times \cdots \times A \rightarrow R$ a nondegenerate symmetric d -linear form on A and let the trace $T : A \rightarrow R$ be defined as above. If θ is invariant under all left and right multiplications corresponding to elements of trace zero, then A is semiprime; i.e., $I^2 \neq 0$ for each non-zero ideal I in A .*

For $d > 2$ the proof can be found in the proof of [S3, Theorem 1 (c)]. A similar argument holds for $d = 2$.

Corollary 1. *Suppose that R is a domain and A an R -algebra with be a nondegenerate form $N : A \rightarrow R$ of degree $d \geq 2$ permitting composition. Then A is a semiprime alternative algebra over R . If I is a minimal ideal of A and $I^2 \neq (0)$, then I is simple and $I^2 = I$. Moreover, I is either a simple associative ring or an octonion algebra over its center. In the latter case, $A = I \oplus I'$ holds as an ideal direct sum.*

Proof. The first assertion follows immediately from Theorem 1 using that the d -linear form associated with N is invariant under all left and right multiplications corresponding to elements of trace zero, the second from [Sl2, Theorem A, Lemma 2.1, Theorem B]. \square

Remark 3. Suppose $d \geq 2$ and that the ring A satisfies the descending chain condition on right ideals. By Corollary 1, A is a semiprime alternative algebra over R . By [Sl1, Theorem B], A is expressible, unique up to order, as an ideal direct sum $A = A_1 \oplus \cdots \oplus A_r$ of minimal ideals A_i where each A_i is either an octonion algebra over its center or a simple artinian associative ring. If every non-zero ideal of A contains a minimal ideal of A and A is purely alternative (i.e., has no non-zero nuclear ideals), then A is expressible, unique up to order,

as an ideal direct sum $A = A_1 \oplus \cdots \oplus A_r$ where each A_i is an octonion algebra over its center [S12, Theorem A].

Lemma 1. (i) $N(P)$ is nondegenerate for all $P \in \text{Spec } R$.

(ii) If A is an associative algebra then A is separable. If A is also central then it is an Azumaya algebra over R .

Proof. (i) Since N is nondegenerate, so is T and thus also $T(P)$ for all $P \in X$. The fact that $T(P)$ is nondegenerate is equivalent to $N(P)$ being nondegenerate ([B, Satz 3] or [B-B, p. 327] for cubic forms) implying that $N(P)$ is a nondegenerate form for all $P \in X$.

(ii) A is finitely generated as R -module by assumption, thus A is separable over R if and only if A/mA is a separable R/m -algebra for all $m \in \text{Max } R$ [Kn1, III. (5.1.10)]. This holds, since $A(P)$ admits a nondegenerate form of degree d permitting composition for all $P \in \text{Spec } R$. Thus $A(P)$ is a finite dimensional separable $k(P)$ -algebra for all $P \in \text{Spec } R$ by [S1, Theorem 3]. Hence A is a separable associative R -algebra, finitely generated as an R -module. If A is central, then A is central, separable and finitely generated faithfully projective as an R -module, therefore an Azumaya algebra by [Kn1, III. (5.1.1)]. \square

Thus, if N permits composition, by (i) the two different notions of nondegeneracy introduced in 2.3 are equivalent.

Remark 4. Assume A has center R' larger than R in the situation of Lemma 1 (ii). Then R' is a separable ring extension of R . View A as an R' -algebra. A is finitely generated and faithful as an R' -module. In case A is projective and separable also as an R' -module, A is an Azumaya algebra over R' [Kn1, III, (5.1.1)].

Proposition 1. $A(P)$ is a finite-dimensional separable algebra over $k(P)$ and

(i) if N is a cubic form, then A has rank 1, 2, 3, 5 or 9 over R ;

(ii) if N is a quartic form, then A has rank 1, 2, 3, 4, 5, 6, 8, 9, 10, 12 or 16 over R .

Proof. For all $P \in \text{Spec } R$ the residue class algebra $A(P)$ is a non-zero alternative algebra over the residue class field $k(P)$ together with a nondegenerate form $N(P) : A(P) \rightarrow k(P)$ which permits composition. Therefore $A(P)$ is a finite-dimensional separable algebra over $k(P)$ [S1, Theorem 3]. Hence, if N is a cubic form, then A must have rank 1, 2, 3, 5 or 9, and if it is a quartic form then A must have rank 1, 2, 3, 4, 5, 6, 8, 9, 10, 12 or 16 [S2, S3]. \square

Corresponding statements can be derived for forms of higher degree than 4.

3.1. For this section we assume that R is a domain.

Lemma 2. Suppose that A contains an idempotent $e \neq 0, 1$. Then 1 and e are linearly independent over R .

Proof. Let $1a + eb = 0$ for $a, b \in R$. Multiplication with e implies $(a + b)e = 0$ and hence $a + b = 0$, since A is a projective R -module, thus torsion free as an R -module. Now $b = -a$, so $0 = 1a + eb = (1 - e)a$ yields $a = 0$ since $1 - e \neq 0$, hence also $b = 0$. \square

Suppose that A contains an idempotent $e \neq 0, 1$. Then

$$x^d - T_1(x)x^{d-1} + T_2(x)x^{d-2} - \cdots + (-1)^d T_d(x)1 = 0$$

for all $x \in A$ implies that

$$1 - T(e) + T_2(e) - \cdots + (-1)^{d-1} T_{d-1}(e)1 = 0,$$

and $N(e) = 0$, since 1 and e are linearly independent. As in [S3, p. 785] we obtain the equations

$$(j+1)T_{j+1}(e) = (T(e) - j)T_j(e)$$

for $j = 1, \dots, d-1$ [S3, (38)], and

$$T_i(e) = \binom{m}{i}$$

for $i = 1, \dots, m$ [S3, (40)], where m is the least integer such that $T_{m+1}(e) = \cdots = T_d(e) = 0$.

These are needed for the proof of the next result, which closely follows the one given in [S3] for Theorem 4.

Theorem 2. *If*

$$A = A_1 \oplus \cdots \oplus A_r$$

is the direct sum of ideals $A_i \neq 0$ in A , then

$$N(x) = N_1(x_1) \cdots N_r(x_r)$$

where $x = x_1 + \cdots + x_r$, $x_i \in A_i$. Each N_i is a nondegenerate form of degree d_i on A_i which permits composition and $d = d_1 + \cdots + d_r$. If $r \geq 2$ then N is absolutely indecomposable.

Proof. Assume that $A = G \oplus G'$ with $G \neq 0$, $G' \neq 0$ ideals. Write $1 = e + e'$ with $e \in G$ and $e' \in G'$. Then $e \neq 0, 1$ (resp. $e' \neq 0, 1$) is the unit element of G (resp. of G') and as such it is an idempotent in A . For $g \in G$, $g' \in G'$ define

$$N_G(g) = N(g + e), \quad N_{G'}(g') = N(e + g').$$

For any $x = g + g' \in A$ we get

$$N(x) = N_G(g)N_{G'}(g')$$

and

$$N_G(g_1 g_2) = N_G(g_1)N_G(g_2)$$

for all $g_1, g_2 \in G$. By showing that

$$N_G(g) = T_m(g) \text{ for all } g \in G, \text{ where } m = T(e),$$

N_G is proved to be a form of degree m over G . (The fact that 1 and e are linearly independent is needed to obtain $T(e) = m$.) Symmetrically, the same formulas hold for $N_{G'}$, so that $N(x)$ is a product of forms of degree m and m' permitting composition and $d = m + m'$. The proof that $N_G(g) = T_m(g)$ for all $g \in G$, with $m = T(e)$ is the same as given in [S3]. Both N_G and $N_{G'}$ are nondegenerate, see the proof in [S3, p. 787, 788].

This argument can be repeated finitely often and since $A = A_1 \oplus \cdots \oplus A_r$ with $A_i \neq 0$ ideals, we obtain the assertion by induction.

That N is absolutely indecomposable follows from [Pr, 5.1]. \square

If R is a field, it is well-known that any form on a simple alternative algebra which permits composition is a power of the generic norm of the algebra. If R is a ring, it can happen that an algebra admits more than one nondegenerate form permitting composition, see [M2].

Remark 5. (i) Let $A = A_1 \oplus A_2$ be a direct sum of two ideals. If A is finitely generated projective as an R -module then so are A_1 and A_2 .
(ii) If $d = 3$ (resp. 4) in Theorem 2, then A is the direct sum of at most three (resp. four) non-zero ideals which all are unital nonassociative algebras admitting a form of degree 1 or 2 (resp. 1, 2 or 3) permitting composition.

4. CUBIC AND QUARTIC FORMS

A nondegenerate cubic (or quartic) form $N : A \rightarrow R$ permitting composition can only exist in rank 1, 2, 3, 5 or 9 (or in rank 1, 2, 3, 4, 5, 6, 8, 9, 10, 12 or 16) by Proposition 1. [S3, Lemma 1, 2, 3] applied to the residue class algebras imply the next two lemmas:

Lemma 3. *Suppose that there exists a cubic form $N : A \rightarrow R$ permitting composition.*

- (i) *If A is a non-split quadratic étale algebra over R , then the residue class algebras $A(P)$ must be split for all $P \in \text{Spec } R$.*
- (ii) *A cannot be a quaternion or octonion algebra over R .*

Lemma 4. *Let A be a composition algebra over R of constant rank with norm n_A . Suppose that there exists a quartic form $N : A \rightarrow R$ permitting composition.*

- (i) *If A is a quadratic étale algebra and $A(P)$ is a quadratic field extension for all $P \in \text{Spec } R$, or if A has rank ≥ 4 then $N(P)(x) = n_{A(P)}(x)^2$ for all $P \in \text{Spec } R$ and $x \in A(P)$.*
- (ii) *A cannot be an Azumaya algebra over R of rank 9.*
- (iii) *If A is a cubic ring extension of R , then $A(P)$ is not a cubic field extension of $k(P)$ for all $P \in \text{Spec } R$.*

Theorem 3. (i) *Let A be an R -algebra of constant rank such that there exists a cubic form N on A permitting composition. Suppose that for each $P \in \text{Spec } R$ there exists an element $u \in A \otimes_R k(P)$ such that $1, u, u^2$ are linearly independent over $k(P)$ (by Theorem 2, A is alternative, so the powers of A are unambiguous). Let M , Q and L be a cubic, a quadratic and a linear form from A to R satisfying*

$$x^3 - L(x)x^2 + Q(x)x - M(x)1 = 0$$

for all $x \in A$. Then $M = N$, $S = T_2$ and $L = T$.

(ii) *Let A be an R -algebra of constant rank such that there exists a quartic form N on A permitting composition. Suppose that for each $P \in \text{Spec } R$ there exists an element $u \in A \otimes_R k(P)$ such that $1, u, u^2, u^3$ are linearly independent over $k(P)$. Let S , M , Q and L be a quartic, cubic, quadratic and a linear form from A to R satisfying*

$$x^4 - L(x)x^3 + Q(x)x^2 - M(x)x + S(x)1 = 0$$

for all $x \in A$. Then $S = N$, $M = T_3$, $Q = T_2$ and $L = T$.

Part (i) is a Corollary of [Ach, 1.12] applied to the Jordan algebra A^+ determined by A , (ii) can be proved analogously (if more tediously). Indeed, both results remain true even if we remove the restriction on R to satisfy $d! \in R^\times$ and work with the more general notion of a form of higher degree as given in [R].

Remark 6. Suppose that A is an Azumaya algebra of constant rank 9 over R . For each $P \in \operatorname{Spec} R$ there exists an element $u \in A(P)$ such that $1, u, u^2$ are linearly independent over $k(P)$, since $A(P)$ is a central simple algebra of degree 3. Since the reduced norm n and trace t of A satisfy $x^3 - t(x)x^2 + q(x)x - n(x)1 = 0$ with q a quadratic form, it follows that $M = n$ (and $L = t$).

Suppose that A is a cubic ring extension of R . For each $P \in \operatorname{Spec} R$ there exists an element $u \in A(P)$ such that $1, u, u^2$ are linearly independent over $k(P)$, since $A(P)$ is a cubic étale algebra over $k(P)$. Since the reduced norm n and trace t of A satisfy $x^3 - t(x)x^2 + q(x)x - n(x)1 = 0$ with q a quadratic form, it follows again that $M = n$ (and $L = t$).

Analogous arguments show that also for a quartic separable ring extension of R and for an Azumaya algebra of constant rank 16 over R , any quartic form on A which permits composition must be uniquely determined and be the norm of the algebra, if $k(P)$ is an infinite field for all $P \in \operatorname{Spec} R$.

For the rest of this section, let R be a domain.

Proposition 2. *Let $N : A \rightarrow R$ be a cubic form on A permitting composition. Then (A, N) is one of the following:*

- (i) $A = R$ and $N(x) = x^3$;
- (ii) A is a separable commutative associative R -algebra of rank 2 or 3, and N is absolutely indecomposable; if A is a non-split quadratic étale algebra over R , then the residue class algebras $A(P)$ must be split for all $P \in \operatorname{Spec} R$, if A has rank 3, then $A(P)$ is a cubic étale algebra over $k(P)$ for all $P \in \operatorname{Spec} R$.
- (iii) A has rank 5 and is a separable associative, but not commutative, R -algebra and N is absolutely indecomposable.
- (iv) If A has rank 9 and is associative, then it is an Azumaya algebra over R and N is its - uniquely determined - norm. If A has rank 9 and is not associative, then A is not commutative and N is absolutely indecomposable.

Proof. Let m denote the rank of A , then $m \in \{1, 2, 3, 5, 9\}$.

- (1) If $m = 1$, then $A = R$ and $N(x) = x^3$.
- (2) If $m = 2$, then $A(P) \cong k(P) \oplus k(P)$ is commutative, associative and $N(P)(x_1 + x_2) = x_1x_2^2$ is absolutely indecomposable for all $P \in \operatorname{Spec} R$. Thus all commutators and associators lie in IA where I is the nil radical of R . Since R is a domain, $I = 0$ and A itself must be a commutative associative R -algebra. $N(P)(x_1 + x_2) = x_1x_2^2$ is absolutely indecomposable for all $P \in \operatorname{Spec} R$, hence N is an absolutely indecomposable form. The rest follows from Lemma 3.

- (3) If $m = 3$, then $A(P)$ is a cubic étale algebra over $k(P)$ and $N(P)$ is its (absolutely indecomposable) norm for all $P \in \text{Spec } R$. Therefore N is absolutely indecomposable and, by the same argument as above, A must be commutative and associative.
- (4) If $m = 5$, then $A(P) \cong k(P) \oplus$ “some quaternion algebra over $k(P)$ ” for all $P \in \text{Spec } R$. Therefore $A(P)$ is associative and not commutative and $N(P)$ is absolutely indecomposable for all $P \in \text{Spec } R$. Thus A is associative by the same argument as above, and not commutative. N is absolutely indecomposable.
- (5) If $m = 9$ we distinguish two cases: If A is associative, then $A(P)$ is a central simple algebra of degree 3 over $k(P)$ for all $P \in \text{Spec } R$ and $N(P)$ is its norm. Therefore A is an Azumaya algebra over R and N is its - uniquely determined - norm. If A is not associative, then $A(P_0) \cong k(P_0) \oplus$ “some octonion algebra over $k(P_0)$ ” for some $P_0 \in \text{Spec } R$ and $N(P_0)$ is absolutely indecomposable. Thus A is not associative and not commutative and N is again absolutely indecomposable.

□

Example 1. Over rings, the first Tits construction [P-R, Theorem 3.5] starting with R can be generalized as follows [Ach, 2.25]: Let $L \in {}_3\text{Pic} R$ and $N_L : L \rightarrow R$ a nondegenerate cubic form, let $L^\vee = \text{Hom}_R(L, R)$ be its dual and $\langle w, \check{w} \rangle = \check{w}(w)$ the canonical pairing $L \times L^\vee \rightarrow R$. There exists a uniquely determined cubic norm $\check{N}_L : L^\vee \rightarrow R$ and uniquely determined adjoints $\sharp : L \rightarrow L^\vee$ and $\check{\sharp} : L^\vee \rightarrow L$ such that

- (1) $\langle w, w^\sharp \rangle = N_L(w)1$;
- (2) $\langle \check{w}^\sharp, \check{w} \rangle = \check{N}_L(\check{w})1$;
- (3) $w^\sharp \check{\sharp} = N_L(w)w$;

for w in L , \check{w} in L^\vee . Moreover,

- (1) $\check{w}^\sharp \check{\sharp} = \check{N}_L(\check{w})\check{w}$;
- (2) $\langle w, \check{w} \rangle^2 = \langle \check{w}^\sharp, w^\sharp \rangle$;
- (3) $\langle w, \check{w} \rangle^3 = N_L(w)\check{N}_L(\check{w})$;
- (4) $\langle w, \check{w} \rangle w = 3\langle w, \check{w} \rangle w - w^\sharp \check{\sharp} \check{w}$

for w, w' in L , \check{w} in L^\vee [Ach, 2.13]. Define

$$\begin{aligned} \tilde{\mathcal{J}} &= R \oplus L \oplus L^\vee, \\ \tilde{1} &= (1, 0, 0), \\ \tilde{N}(a, w, \check{w}) &= a^3 + N_L(w) + \check{N}_L(\check{w}) - 3a\langle w, \check{w} \rangle \\ (a, w, \check{w})^\sharp &= (a^2 - \langle w, \check{w} \rangle, \check{w}^\sharp - aw, w^\sharp - \check{w}a) \end{aligned}$$

for $a \in A$, $w \in L$, $\check{w} \in L^\vee$, then $(\tilde{N}, \sharp, \tilde{1})$ is a cubic form with adjoint and base point on $\tilde{\mathcal{J}}$ and has trace form

$$\tilde{T}((a, w, \check{w}), (c, v, \check{v})) = 3ac + 3\langle w, \check{v} \rangle + 3\langle v, \check{w} \rangle.$$

The Jordan algebra $A = \mathcal{J}(\tilde{N}, \sharp, \tilde{1})$ over R is commutative and associative and the nondegenerate cubic form \tilde{N} permits composition. $A(P)$ is a cubic étale algebra over $k(P)$ for all $P \in \text{Spec } R$.

Lemma 5. *Let $N : A \rightarrow R$ be a nondegenerate cubic form on A permitting composition.*

(i) *Suppose A can be written as the direct sum of two non-zero ideals A_1, A_2 of A . Then $A = R \oplus A_2$ where A_2 is a composition algebra over R with norm n and $N(x_1 + x_2) = x_1 n(x_2)$ is absolutely indecomposable.*

(ii) *Suppose A can be written as the direct sum of three non-zero ideals A_1, A_2, A_3 of A . Then $A = R \oplus R \oplus R$ and $N(x_1 + x_2 + x_3) = x_1 x_2 x_3$.*

Proof. (1) Suppose A can be written as the direct sum of two non-zero ideals. Then $A = A_1 \oplus A_2$ and $N(x_1 + x_2) = n_1(x_1)n(x_2)$ with $n_1(x_1) = x_1$ and n a nondegenerate quadratic form permitting composition. Therefore $A = R \oplus A_2$ where A_2 is an algebra over R with a nondegenerate quadratic form n permitting composition and $N(x_1 + x_2) = x_1 n(x_2)$ is absolutely indecomposable. Since R is a domain, A_2 has full support, hence it is a composition algebra over R .

(2) Suppose A can be written as the direct sum of three non-zero ideals $A = A_1 \oplus A_2 \oplus A_3$. Then $N(x_1 + x_2 + x_3) = n_1(x_1)n_2(x_2)n_3(x_3)$ with n_i linear forms permitting composition, $i = 1, 2, 3$. Hence $A = R \oplus R \oplus R$ and $N(x_1 + x_2 + x_3) = x_1 x_2 x_3$. \square

From now on suppose that $k(P)$ is an infinite field for all $P \in \text{Spec } R$.

Proposition 3. *Let $N : A \rightarrow R$ be a quartic form on A permitting composition. Then one of the following holds:*

- (i) $A = R$ and $N(x) = x^4$.
- (ii) A is a commutative associative separable R -algebra of rank 2 or 3. If A has rank 3, then $A(P)$ cannot be a cubic field extension of $k(P)$ for all $P \in \text{Spec } R$.
- (iii) A has rank 4 and is an associative separable R -algebra. If, in particular, A is not commutative, then $A(P_0)$ is a quaternion algebra over $k(P_0)$ for at least one P_0 .
- (iv) A has rank 5 or 6 and A is an associative separable R -algebra, but not commutative.
- (v) A has rank 8 and A is not commutative. If A is not associative, then $A(P_0)$ is an octonion algebra over $k(P_0)$ for some $P_0 \in \text{Spec } R$.
- (vi) A has rank 9, 10 or 12, is not associative, not commutative.
- (vii) A has rank 16 and A is not associative. Then $A(P_0)$ is an octonion algebra over some quadratic field extension of $k(P_0)$, for some $P_0 \in \text{Spec } R$.
- (viii) A has rank 16 and A is an Azumaya algebra over R with norm N .

In cases (i) to (vii), N is absolutely indecomposable.

Proof. Let m be the rank of A , then $m \in \{1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 16\}$.

- (1) If $m = 1$, then $A = R$ and $N(x) = x^4$.
- (2) If $m = 2$, then $A(P)$ is isomorphic to a quadratic étale algebra over $k(P)$ and $N(P) = (x_1 + x_2)^2 = x_1^2 x_2^2$ is absolutely indecomposable for all $P \in \text{Spec } R$, implying that N must be absolutely indecomposable. By the same argument as used in the proof of Proposition 2 (2), A is a commutative associative R -algebra together.
- (3) If $m = 3$, then $A(P) \cong k(P) \oplus$ “some quadratic étale algebra over $k(P)$ ” by [S3, Lemma 2] and $N(P)$ is absolutely indecomposable for all $P \in \text{Spec } R$. Thus A is commutative, associative and separable and N absolutely indecomposable.

- (4) If $m = 4$, then $A(P)$ is either a quaternion algebra over $k(P)$, a separable quartic field extension over $k(P)$, a quadratic étale algebra over some quadratic field extension of $k(P)$ or the direct sum of two quadratic étale algebras over $k(P)$. Hence A is associative and separable. In particular, if A is not commutative, then $A(P_0)$ must be a quaternion algebra over R for a $P_0 \in \text{Spec } R$.
- (5) If $m = 5$, then $A(P) \cong k(P) \oplus$ “some quaternion algebra over $k(P)$ ” for all $P \in \text{Spec } R$ and $N(P)$ is absolutely indecomposable. Thus A is associative and separable, but not commutative, and N absolutely indecomposable.
- (6) If $m = 6$ then $A(P)$ is isomorphic to the direct sum of a quadratic étale algebra and some quaternion algebra for all $P \in \text{Spec } R$. Thus A is associative and separable, not commutative, and N absolutely indecomposable.
- (7) If $m = 8$ then $A(P)$ is either an octonion algebra over $k(P)$, a quaternion algebra over some quadratic field extension of $k(P)$, or the direct sum of two quaternion algebras over $k(P)$. Hence A is not commutative and N absolutely indecomposable. If, in particular, A is not associative, then $A(P_0)$ is an octonion algebra over $k(P_0)$ for some $P_0 \in \text{Spec } R$ and $N(P_0)$ is the square of its norm.
- (8) If $m = 9$, then $A(P) \cong k(P) \oplus$ “some octonion algebra over $k(P)$ ” for all $P \in \text{Spec } R$. Thus A is not associative, not commutative, and N absolutely indecomposable.
- (9) If $m = 10$, then $A(P)$ is isomorphic to the direct sum of a quadratic étale algebra and some octonion algebra over $k(P)$, or $A(P) \cong k(P) \oplus$ “some Azumaya algebra of degree 3 over $k(P)$ ”. Thus A is not associative, not commutative and N absolutely indecomposable.
- (10) If $m = 12$, then $A(P)$ is isomorphic to the direct sum of some octonion algebra and some quaternion algebra for all $P \in \text{Spec } R$. Again A is not commutative, not associative and N absolutely indecomposable.
- (11) If $m = 16$ and A is associative, then $A(P)$ is a central simple algebra over $k(P)$ for all $P \in \text{Spec } R$. Hence A is an Azumaya algebra over R with norm N . If A is not associative, then $A(P_0)$ is an octonion algebra over some quadratic field extension of $k(P_0)$, for some $P_0 \in \text{Spec } R$.

□

Lemma 6. *Let $N : A \rightarrow R$ be a nondegenerate quartic form N on A permitting composition.*

- (1) *Suppose A can be written as the direct sum of two non-zero ideals. Then $A = A_1 \oplus A_2$ and $N(x_1 + x_2) = n_1(x_1)n_2(x_2)$. Either A_1 and A_2 are composition algebras and n_1 and n_2 are their norms, or $A_1 = R$, $n_1(x_1) = x_1$ and n_2 is a nondegenerate cubic form permitting composition.*
- (2) *Suppose A can be written as the direct sum of three non-zero ideals. Then $A \cong R \oplus R \oplus A_3$ and $N(x_1 + x_2 + x_3) = x_1x_2n_3(x_3)$ with A_3 a composition algebra over R with norm n_3 .*
- (3) *Suppose A can be written as the direct sum of four non-zero ideals. Then $A \cong R \oplus R \oplus R \oplus R$ and $N(x_1 + x_2 + x_3 + x_4) = x_1x_2x_3x_4$.*

Each N is absolutely indecomposable.

Proof. (1) Suppose A can be written as the direct sum of two non-zero ideals. Then $A = A_1 \oplus A_2$ and $N(x_1 + x_2) = n_1(x_1)n_2(x_2)$ with n_1 and n_2 being either two nondegenerate quadratic forms permitting composition, or $A_1 = R$, $n_1(x_1) = x_1$ and n_2 is a nondegenerate cubic form permitting composition (Theorem 4 (i)). Since R is a domain, A_1 and A_2 have full support in both cases.

(2) Suppose A can be written as the direct sum of three non-zero ideals. Then $A \cong A_1 \oplus A_2 \oplus A_3$ with $A_1 = A_2 = R$, and $N(x_1 + x_2 + x_3) = n_1(x_1)n_2(x_2)n_3(x_3)$, $n_1(x_1) = x_1$, $n_2(x_2) = x_2$ and n_3 a nondegenerate quadratic form permitting composition (Theorem 4 (i)). Thus A_3 is a composition algebra over R , since R is a domain.

The rest of the assertion is clear. \square

5. EXAMPLES

Let k be an infinite field. Let A be a unital nonassociative algebra of constant rank over R which is finitely generated projective and faithful as R -module. Let $\text{Cay}(D, \mu)$ be the classical Cayley-Dickson doubling of the composition algebra D with scalar $\mu \in R^\times$ (cf. for instance [P]).

5.1. Let $R = k[t]$ be the polynomial ring over k . Suppose that k has characteristic 0 or greater than 3 and that there exists a nondegenerate cubic form on A permitting composition. If A is the direct sum of two non-zero ideals, then $A = R \oplus C$ and $N(x_1 + x_2) = x_1 n_C(x_2)$, where $C = C_0 \otimes R$ is a composition algebra defined over k with norm n_C [P, 6.8]. Hence both $A = R \oplus C$ and N are defined over k and N is absolutely indecomposable.

If $n_{l/k}$ is the norm of a cubic field extension of k , then $N = n_{l/k} \otimes_k R$ is another example of an indecomposable cubic form over R permitting composition, again N is defined over k .

Suppose now that k has characteristic 0 or greater than 4 and that there exists a nondegenerate quartic form on A permitting composition. The following are examples of such an (A, N) :

- (i) $A = R$ and $N(x) = x^4$;
- (ii) A composition algebra A over R with norm n and $N(x) = n(x)^2$. Then both A and N are defined over k and N is absolutely indecomposable.
- (iii) A separable quartic ring extension A of R and N its norm, e.g. $N = n_{l/k} \otimes_k R$ where $n_{l/k}$ is the norm of a separable quartic field extension of k .
- (iii) A composition algebra A of constant rank 4 or 8 over its center, which is a separable quadratic ring extension R' of R . A has a (unique) norm $n_{A/R'}$ and $N(x) = n_{R'/R}(n_{A/R'}(x))$. Thus R' is isomorphic to $R \oplus R$ or to $k(\sqrt{c}) \otimes_k R$. Say $R' = k(\sqrt{c}) \otimes_k R \cong k(\sqrt{c})[t]$. Then either A is isomorphic to $\text{Mat}_2(R')$ or Z or R' , or it is without zero divisors and defined over $k(\sqrt{c})$.
- (iv) An Azumaya algebra A of rank 16 over R and N its norm.
- (v) $A = C \oplus D$ and $N(x_1 + x_2) = n_C(x_1)n_D(x_2)$ with C, D two composition algebras over R which are defined over k . A and N are defined over k with norms n_C, n_D . N is absolutely indecomposable.
- (vi) $A = R \oplus A_2$, where A_2 is an algebra over R with a cubic form n_2 permitting composition; that means, N is absolutely indecomposable. For instance,

- (1) $A = R \oplus R$ and $N(x_1 + x_2) = x_1 x_2^3$;
- (2) $A = R \oplus R \oplus A_2$ and $N(x_1 + x_2 + x_3) = x_1 x_2 n(x_3)$, where A_2 is a cubic étale or an Azumaya algebra of rank 9 over R and n its norm;
- (3) $A = R \oplus R \oplus C$ and $N(x_1 + x_2 + x_3) = x_1 x_2 n_C(x_3)$, where C is a composition algebra over R defined over k with norm n_C . A and N are defined over k .

(vii) $A = R \oplus R \oplus C$ and $N(x_1 + x_2 + x_3) = x_1 x_2 n_C(x_3)$, where C is a composition algebra over R . Both A and N are defined over k , N is absolutely indecomposable.

5.2. Let $R = k[t, \frac{1}{t}]$ be the ring of Laurent polynomials over k . Suppose that k has characteristic 0 or greater than 3 and that there exists a nondegenerate cubic form on A permitting composition. If A is the direct sum of two non-zero ideals, then $A = R \oplus C$ where C is a composition algebra over R with norm n_C and $N(x_1 + x_2) = x_1 n_C(x_2)$. Thus N is absolutely indecomposable and either C (hence A and N) is defined over k or it is isomorphic to $\text{Cay}(D, \mu t)$, where D is a composition algebra without zero divisors of half the rank of C which is defined over k , and $\mu \in k^\times$ arbitrary [Pu1].

Other examples of absolutely indecomposable cubic forms over R permitting composition are of the type $N = n_{l/k} \otimes_k R$ where $n_{l/k}$ is the norm of a cubic field extension of k , or of the kind $A = J(R, \mu t)$ and $N(u, v, w) = u^3 + \mu t v^3 + \mu^{-1} t^{-1} w^3 - 3 \mu t u v w$ where $J(R, \mu t)$ is the first Tits construction starting with R , $\mu \in k^\times$; i.e., $J(R, \mu t) = R(\sqrt[3]{\mu t})$, cf. Example 1.

Suppose now that k has characteristic 0 or greater than 4 and that there exists a nondegenerate quartic form on A permitting composition. The following are examples of such (A, N) :

- (i) $A = R$ and $N(x) = x^4$.
- (ii) A a composition algebra over R with norm n and $N(x) = n(x)^2$. Then A is split and isomorphic to $R \oplus R$, $\text{Mat}_2(R)$ or $\text{Zor } R$, or $A = A_0 \otimes_k R$ with A_0 a composition division algebra over k . Consider the non-split case: If it has rank 2 it is either isomorphic to $k(\sqrt{c}) \otimes_k R$ for some $c \in k^\times$ which is not a square, or to $\text{Cay}(R, \mu t)$ with $\mu \in k^\times$. Every composition algebra of rank greater than 2 without zero divisors is either defined over k or it is isomorphic to $\text{Cay}(T, \mu t)$ where T is a composition algebra without zero divisors of half the rank which is defined over k , and $\mu \in k^\times$ arbitrary.
- (iii) A separable quartic ring extension of R with norm N .
- (iv) A composition algebra of constant rank 4 or 8 with center R' , which is a separable quadratic ring extension of R with (unique) norm $n_{A/R'}$ and $N(x) = n_{R'/R}(n_{A/R'}(x))$.

In particular, it is possible that R' is isomorphic to $k(\sqrt{c}) \otimes_k R$ or to $\text{Cay}(R, \mu t)$ with $\mu \in k^\times$ (unless R' is not étale, in that case there might be others). Suppose that $R' = k(\sqrt{c}) \otimes_k R \cong k(\sqrt{c})[t, \frac{1}{t}]$ for some quadratic field extension $k(\sqrt{c})$ of k . Then A is isomorphic to $\text{Mat}_2(R')$ or $\text{Zor } R'$, or it is a composition division algebra over R' . It is either defined over $k(\sqrt{c})$ or it is isomorphic to $\text{Cay}(T, \mu t)$ where T is a composition algebra without zero divisors of half the rank which is defined over $k(\sqrt{c})$, and $\mu \in k(\sqrt{c})^\times$ arbitrary.

- (iv) An Azumaya algebra of rank 16 over R with norm N .
- (v) $A = C \oplus D$ and $N(x_1 + x_2) = n_C(x_1) n_D(x_2)$ with C, D two composition algebras over R which are defined over k . N is absolutely indecomposable.

(vi) $A = R \oplus A_2$, where A_2 is an algebra over R with a cubic form n_2 permitting composition; so N is absolutely indecomposable.

(vii) $A = R \oplus R \oplus C$ and $N(x_1 + x_2 + x_3) = x_1 x_2 n_C(x_3)$, where C is a composition algebra over R with norm n_C ; i.e., C is defined over k or isomorphic to $\text{Cay}(T, \mu)$ where T is a composition algebra without zero divisors of half the rank which is defined over k , $\mu \in k^\times$ arbitrary. N is absolutely indecomposable.

6. FORMS PERMITTING COMPOSITION OVER LOCALLY RINGED SPACES

Let X be a locally ringed space with structure sheaf \mathcal{O}_X such that $d! \in H^0(X, \mathcal{O}_X^\times)$.

An \mathcal{O}_X -algebra \mathcal{A} is called *alternative* if $x^2 y = x(xy)$ and $yx^2 = (yx)x$ for all sections x, y of \mathcal{A} over the same open subset of X . An associative \mathcal{O}_X -algebra \mathcal{A} is called an *Azumaya algebra* if $\mathcal{A}_P \otimes_{\mathcal{O}_{P,X}} k(P)$ is a central simple algebra over $k(P)$ for all $P \in X$.

Let \mathcal{A} be an \mathcal{O}_X -algebra together with a nondegenerate form $N : \mathcal{A} \rightarrow \mathcal{O}_X$ of degree d permitting composition; i.e., $N(xy) = N(x)N(y)$ for all sections x, y of \mathcal{A} over the same open subset of X . Let $1 = 1_{\mathcal{A}} \in H^0(X, \mathcal{A})$ be the unit element of \mathcal{A} . Suppose always that $N(1) = 1$. Then \mathcal{A} has full support.

Remark 7. (i) Let $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$ be the direct sum of two non-zero ideals of \mathcal{A} . Since \mathcal{A} is locally free of finite rank as \mathcal{O}_X -module by our convention 2.4, so are \mathcal{A}_1 and \mathcal{A}_2 .

(ii) Let X be a scheme over the affine scheme $Y = \text{Spec } R$, $H^0(X, \mathcal{O}_X) = R$. If \mathcal{A} is globally free as an \mathcal{O}_X -module then (\mathcal{A}, N) is defined over R . (The proof is analogous to the one of [P, 1.10].)

Let $\theta : \mathcal{A} \times \cdots \times \mathcal{A} \rightarrow \mathcal{O}_X$ be the d -linear form associated with N . For $i = 1, \dots, d$ define a form $T_i : \mathcal{A} \rightarrow \mathcal{O}_X$ of degree i via

$$T_i(x) = \binom{n}{i} \theta(x, \dots, x, 1, \dots, 1) \quad (i\text{-times } x).$$

Then

$$N(x) = T_d(x) \text{ and } T_1(x) = d\theta(x, 1, \dots, 1)$$

for all sections x of \mathcal{A} over the same open subset of X .

The form $T : \mathcal{A} \rightarrow \mathcal{O}_X$, $T(x) = T_1(x)$ is called the *trace*. Put $T_0(x) = 1$ and $T_{d+q}(x) = 0$ for $q > 0$, then $T(a1) = da\theta(1, \dots, 1)$ for all a in \mathcal{O}_X . Define $\mathcal{A}_0 = \ker T$. \mathcal{A} is the direct sum of \mathcal{O}_X -modules $\mathcal{A} = \mathcal{O}_X 1_{\mathcal{A}} \oplus \mathcal{A}_0$.

We assume from now on that $d \geq 3$. Our results from Section 3 easily adapt to the setting of locally ringed spaces: \mathcal{A} is alternative and $B : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{O}_X$, with $B(x, y) = T(xy)$ for all sections x, y of \mathcal{A} over the same open subset of X , is a nondegenerate symmetric bilinear form on \mathcal{A} which is associative. Every section x of \mathcal{A} over the same open subset of X satisfies

$$x^d - T_1(x)x^{d-1} + T_2(x)x^{d-2} - \cdots + (-1)^d T_d(x)1 = 0.$$

From now on let X be an integral scheme. If $\mathcal{A} = \mathcal{A}_1 \oplus \cdots \oplus \mathcal{A}_r$ with \mathcal{A}_i non-zero ideals of \mathcal{A} , then

$$N(x) = N_1(x_1) \cdots N_r(x_r)$$

where $x = x_1 + \cdots + x_r$, $x_i \in \mathcal{A}_i$ for all i , and each N_i is a nondegenerate form of degree d_i on \mathcal{A}_i , $d = d_1 + \cdots + d_r$, which permits composition.

Lemma 7. (i) $\mathcal{A}(P)$ is separable for all $P \in X$.

(ii) If \mathcal{A} is not associative, but $\mathcal{A}(P)$ is simple for all $P \in X$, then there is at least one $P \in X$ such that $\mathcal{A}(P)$ is an octonion algebra over some separable field extension of $k(P)$.

Proof. (i) We know that for all $P \in X$, $\mathcal{A}(P)$ is a non-zero $k(P)$ -algebra such that there exists a nondegenerate form $N(P)$ of degree d on $\mathcal{A}(P)$ permitting composition (2.3 and Lemma 1). Thus $\mathcal{A}(P)$ is a finite dimensional separable $k(P)$ -algebra for all $P \in X$.

(ii) follows immediately. \square

Proposition 4. (i) If N is a cubic form then \mathcal{A} has rank 1, 2, 3, 5 or 9.

(ii) If N is a quartic form, then \mathcal{A} has rank 1, 2, 3, 4, 5, 6, 8, 9, 10, 12 or 16.

Proof. For all $P \in \text{Spec} R$ the residue class algebra $\mathcal{A}(p) = \mathcal{A}_P \otimes k(P)$ is a nonzero alternative algebra over the residue class field $k(P)$ together with a nondegenerate form $N(P) : \mathcal{A}(P) \rightarrow k(P)$ which permits composition (2.3 and Lemma 1). This implies that if N is a cubic form then $\mathcal{A}(P)$ must have rank 1, 2, 3, 5 or 9, and if it is a quartic form then $\mathcal{A}(P)$ must have rank 1, 2, 3, 4, 5, 6, 8, 9, 10, 12 or 16 [S1, p. 140]. \square

We now turn to cubic forms permitting composition.

Lemma 8. A nondegenerate cubic form N permitting composition cannot be defined

(i) on a composition algebra over X of constant rank greater than 2;

(ii) on a non-split quadratic étale algebra \mathcal{A} over X , unless $\mathcal{A}(P)$ is split for all $P \in X$.

This can be proved by assuming such a form exists and using Lemma 3 locally to obtain a contradiction.

Proposition 5. Suppose that there exists a nondegenerate cubic form N on \mathcal{A} permitting composition. Then (\mathcal{A}, N) satisfies one of the following:

(i) $\mathcal{A} = \mathcal{O}_X$ and $N(x) = x^3$;

(ii) \mathcal{A} is commutative and associative of rank 2 or 3, and N is absolutely indecomposable; if \mathcal{A} is a non-split quadratic étale algebra over X , then $\mathcal{A}(P)$ must be split for all $P \in X$; if \mathcal{A} has rank 3 then $\mathcal{A}(P)$ is a cubic étale algebra over $k(P)$ for all $P \in X$.

(iii) \mathcal{A} is an associative and not commutative \mathcal{O}_X -algebra of rank 5, and N is absolutely indecomposable.

(iv) \mathcal{A} is an Azumaya algebra over X of rank 9, and N is its reduced norm.

(v) \mathcal{A} has rank 9 and is neither associative nor commutative, N is absolutely indecomposable.

This follows from Proposition 2.

Example 2. There is the following first Tits construction starting with the structure sheaf of an integral scheme X [Ach, 2.25]: Let $\mathcal{L} \in {}_3\text{Pic} X$ and $N : \mathcal{L} \rightarrow \mathcal{O}_X$ a nondegenerate cubic form, let $\check{\mathcal{L}} = \text{Hom}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$ be its dual and $\langle w, \check{w} \rangle = \check{w}(w)$ the canonical pairing $\mathcal{L} \times \check{\mathcal{L}} \rightarrow \mathcal{O}_X$. There exists a uniquely determined cubic norm $\check{N} : \check{\mathcal{L}} \rightarrow \mathcal{O}_X$ and uniquely

determined adjoints $\sharp : \mathcal{L} \rightarrow \check{\mathcal{L}}$ and $\check{\sharp} : \check{\mathcal{L}} \rightarrow \mathcal{L}$ such that the identities listed in Example 1 (with N instead of N_L) hold for w, w' in \mathcal{L} , \check{w} in $\check{\mathcal{L}}$ [Ach, 2.13]. Define

$$\begin{aligned}\tilde{\mathcal{J}} &= \mathcal{O}_X \oplus \mathcal{L} \oplus \check{\mathcal{L}}, \\ \tilde{1} &= (1, 0, 0) \in H^0(X, \tilde{\mathcal{J}}), \\ \tilde{N}(a, w, \check{w}) &= a^3 + N(w) + \check{N}(\check{w}) - 3a\langle w, \check{w} \rangle \\ (a, w, \check{w})^{\tilde{\sharp}} &= (a^2 - \langle w, \check{w} \rangle, \check{w}^{\sharp} - aw, w^{\sharp} - \check{w}a)\end{aligned}$$

for $a \in \mathcal{O}_X$, $w \in \mathcal{L}$, $\check{w} \in \check{\mathcal{L}}$, then $(\tilde{N}, \tilde{\sharp}, \tilde{1})$ is a cubic form with adjoint and base point on $\tilde{\mathcal{J}}$ and has trace form

$$\tilde{T}((a, w, \check{w}), (c, v, \check{v})) = 3ac + 3\langle w, \check{v} \rangle + 3\langle v, \check{w} \rangle.$$

The induced (commutative associative) Jordan algebra $J(\tilde{N}, \tilde{\sharp}, \tilde{1})$ is denoted by $J(\mathcal{A}, \mathcal{P}, N)$. This construction yields examples of commutative associative algebras $\mathcal{A} = \mathcal{J}(\mathcal{O}_X, \mathcal{L}, N)$ of constant rank 3 which admit a nondegenerate cubic form \tilde{N} permitting composition. $\mathcal{A}(P)$ is a cubic étale algebra over $k(P)$ for all $P \in \text{Spec } X$.

Example 3 (Pu2). Let k be a field of characteristic 0. Let X be an elliptic curve X over k . Let \mathcal{N}_i denote a line bundle of order 3 on X with $\mathcal{N}_0 = \mathcal{O}_X$. We have ${}_3\text{Pic}(X) = \{\mathcal{N}_i \mid 0 \leq i \leq 2\}$ for some even integer $0 \leq m \leq 8$. Every first Tits construction over X starting with \mathcal{O}_X is isomorphic to $\mathcal{A} = \mathcal{J}(\mathcal{O}_X, \mathcal{N}_i, N_i)$ where N_i is a nondegenerate cubic form on \mathcal{N}_i . By the Theorem of Krull-Schmidt, if $\mathcal{N}_i \not\cong \mathcal{N}_j$ and $\mathcal{N}_i \not\cong \mathcal{N}_j^\vee$ then $\mathcal{J}(\mathcal{O}_X, \mathcal{N}_i, N_i) \not\cong \mathcal{J}(\mathcal{O}_X, \mathcal{N}_j, N_j)$. $\mathcal{J}(\mathcal{O}_X, \mathcal{N}_0, N_0)$ is defined over k .

7. CURVES OF GENUS ZERO

Lemma 9. *Let X be a curve of genus zero over a field k of characteristic not 2 or 3. Let \mathcal{A} be an algebra over X of rank 2 carrying a nondegenerate form N permitting composition. If N is a cubic or quartic form then (\mathcal{A}, N) is defined over k .*

Proof. \mathcal{A} , together with the nondegenerate symmetric bilinear form $T_{\mathcal{A}}$, is a nondegenerate bilinear space over X . If X is rational, [P, 5.4] shows that \mathcal{A} decomposes into the orthogonal sum of \mathcal{O}_X -modules of the kind $\mathcal{O}_X(m_i) \oplus \mathcal{O}_X(-m_i)$ for suitable $m_i > 0$. Hence, if \mathcal{A} has rank two, it must be globally free as an \mathcal{O}_X -module.

If X is nonrational, there is a field extension k'/k such that $X' = X \times_k k'$ becomes rational. If \mathcal{A} is an algebra as in our assumption then so is $\mathcal{A} \otimes \mathcal{O}_{X'}$ and since this is globally free, so is \mathcal{A} . By Remark 7 (ii), in both cases (\mathcal{A}, N) is defined over k . \square

Lemma 10. *Let X be a curve of genus zero over a field k of characteristic not 2 or 3. Every first Tits construction over X starting with \mathcal{O}_X is defined over k .*

Proof. Every first Tits construction over X starting with \mathcal{O}_X is of the kind $J(\mathcal{O}_X, \mathcal{L}, N_{\mathcal{L}})$, where $\mathcal{L} \in {}_3\text{Pic} X$ (Example 2). However, $\text{Pic } X \cong \mathbb{Z}$, so we only have $J(\mathcal{O}_X, \mathcal{O}_X, \mu) \cong J(k, \mu) \otimes_k \mathcal{O}_X$, $\mu \in k^\times$. \square

7.1. Let k be a field of characteristic 0 and X be a curve of genus zero over k . If X is not rational, let $D_0 = (a, b)_k$ be the quaternion division algebra associated with X . Let \mathcal{A} be an algebra (automatically of constant rank) over X such that there exists a nondegenerate cubic form N on \mathcal{A} permitting composition.

- (1) If \mathcal{A} has rank 1, then $\mathcal{A} = \mathcal{O}_X$ and $N(x) = x^3$.
- (2) If \mathcal{A} has rank 3 it is commutative associative and N is absolutely indecomposable. For instance, let k' be a cubic field extension of k , then $\mathcal{A} = k' \otimes_k \mathcal{O}_X$ carries a nondegenerate cubic form permitting composition.
- (3) If \mathcal{A} is the direct sum of two non-zero ideals $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$, then $N(x_1 + x_2) = N_1(x_1)N_2(x_2)$ and N_1 must be a nondegenerate linear form and N_2 a nondegenerate quadratic one (or vice versa). It follows that $\mathcal{A}_1 = \mathcal{O}_X$, $N_1 = id$, and \mathcal{A}_2 is an algebra of degree 2 over X of constant rank with a nondegenerate quadratic form N_2 permitting composition. Therefore \mathcal{A}_2 is a composition algebra over X of constant rank with norm N_2 . Hence

$$\mathcal{A} = \mathcal{O}_X \oplus \mathcal{C}$$

with \mathcal{C} a quadratic étale algebra, a quaternion or an octonion algebra over X , and N is absolutely indecomposable. (Note that \mathcal{A} cannot be the direct sum of more than two non-zero ideals.) By [P, 4.4], one of the following holds:

- (i) \mathcal{C} (and thus \mathcal{A}) is defined over k .
- (ii) \mathcal{C} is a split quaternion or octonion algebra.
- (iii) X is not rational and $\mathcal{C} \cong \text{Cay}(\mathcal{D}, \mathcal{P}, N_{\mathcal{P}})$, where $\mathcal{D} = D_0 \otimes \mathcal{O}_X$, \mathcal{P} is a locally free right \mathcal{D} -module of rank one and norm one, and $N_{\mathcal{P}}$ is a norm on it. More precisely, let \mathcal{E}_0 be the indecomposable \mathcal{O}_X -module of rank 2 described in [P, 4.3]. Then we know that $\mathcal{P} = \mathcal{P}_1 \oplus \mathcal{P}_2$ with $\mathcal{P}_1 = \mathcal{L}(mP_0) \otimes \check{\mathcal{E}}_0$ and $\mathcal{P}_2 = \mathcal{L}((-m+1)P_0) \otimes \check{\mathcal{E}}_0$ for some integer $m \geq 0$ uniquely determined by \mathcal{C} , where P_0 is a closed point of X of minimal degree.
- (4) If \mathcal{A} is the direct sum of three non-zero ideals $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \mathcal{A}_3$, then $N(x_1 + x_2 + x_3) = N_1(x_1)N_2(x_2)N_3(x_3)$ and N_i must be a nondegenerate linear form ($i = 1, 2, 3$). It follows that $N(x_1 + x_2 + x_3) = x_1x_2x_3$ and \mathcal{A} is defined over k .
- (5) Suppose that \mathcal{A} does not decompose into the direct sum of non-zero ideals. If \mathcal{A} has rank 9 and is associative then \mathcal{A} is an Azumaya algebra over X of rank 9 and N its reduced norm. If X is rational, then either we have

$$\mathcal{A} \cong \text{End}_X(\mathcal{O}_X(m_1) \oplus \mathcal{O}_X(m_2) \oplus \mathcal{O}_X(m_3))$$

with $m_i \in \mathbb{Z}$, or both \mathcal{A} and N are defined over k and we have

$$\mathcal{A} \cong \sigma^* D$$

with D a central simple division algebra over k ([Kn1, VII (3.1.1)], [Kn2], see also [Ach]). If X is nonrational, we can give some examples of Azumaya algebras of rank 9 over X :

For instance, again the trivial case that $\mathcal{A} \cong \sigma^* D$ with D a central simple division algebra over k .

For every locally free \mathcal{O}_X -module \mathcal{E} of constant rank 3, $\mathcal{E}nd_X(\mathcal{E})$ is an Azumaya algebra of rank 9 where we have the following possibilities for \mathcal{E} :

$$\mathcal{E} = \mathcal{L}(m_1 P_0) \otimes \mathcal{E}_0 \oplus \mathcal{L}(m_2 P_0) \text{ and } \mathcal{E} = \mathcal{L}(n_1 P_0) \oplus \mathcal{L}(n_2 P_0) \oplus \mathcal{L}(n_3 P_0)$$

with $m_1, m_2, n_1, n_2, n_3 \in \mathbb{Z}$. Hence

(i)

$$\mathcal{E}nd_X(\mathcal{E}) = \begin{bmatrix} \mathcal{E}nd(\mathcal{E}_0) & \mathcal{L}(-aP_0) \otimes \mathcal{E}_0 \\ \mathcal{L}(aP_0) \otimes \mathcal{E}_0 & \mathcal{O}_X \end{bmatrix}$$

(ii)

$$\mathcal{E}nd_X(\mathcal{E}) = \begin{bmatrix} \mathcal{O}_X & \mathcal{L}(cP_0) & \mathcal{L}(bP_0) \\ \mathcal{L}(-cP_0) & \mathcal{O}_X & \mathcal{L}((b-c)P_0) \\ \mathcal{L}(-bP_0) & \mathcal{L}((c-b)P_0) & \mathcal{O}_X \end{bmatrix}$$

with $a = m_2 - m_1$, $b = n_1 - n_3$ and $c = n_1 - n_2$ [Ach, 4.4].

8. DEGENERATE FORMS PERMITTING COMPOSITION

8.1. Let us consider forms $N : A \rightarrow R$ permitting composition over rings R with $d! \in R^\times$, where A is a unital nonassociative R -algebra which is finitely generated projective as R -module. We now look at the case where $N : A \rightarrow R$ is degenerate; i.e., where the *radical*

$$\text{rad } N = \{x \in A \mid \theta(x, x_2, \dots, x_d) = 0 \text{ for all } x_i \in A\}$$

is non-zero. As before, let $B : A \times A \rightarrow R$, $B(x, y) = T(xy)$ where T is the trace of A .

Lemma 11. *Let A be an R -algebra together with a form N of degree d on A permitting composition. Let D be a subalgebra of A which is maximal among all subalgebras E of A which have a nondegenerate restriction $N|_E$. Then $(A, B) = (D, B|_D) \perp (D^\perp, B|_{D^\perp})$ and $DD^\perp \subset D^\perp$ as well as $D^\perp D \subset D^\perp$.*

Proof. Since N_D is nondegenerate, so is $B|_D$. Thus $(A, B) = (D, B|_D) \perp (D^\perp, B|_{D^\perp})$ by [Kn1, I(3.6.2)]. Let $x \in D$ and $y \in D^\perp$. Then $B(z, xy) = B(zx, y) = 0$ and $B(yx, z) = B(y, xz) = 0$ for all $z \in D$, thus $xy \in D^\perp$ and $yx \in D^\perp$. Since B is an associative symmetric bilinear form, $\text{rad } B$ is a two-sided ideal. We have $\text{rad}(B) = \text{rad}(N)$ by Lemma 1 (i). \square

For $d = 2$ and R a field this was proved in [K-S, 1.2].

Remark 8. The radical $\text{rad } N$ of a cubic form N permitting composition on an R -algebra A is a two-sided ideal [B-B, Lemma 2] (indeed, this is true for the radical of any form of degree d permitting composition [B]). If A is also an algebra of degree 3 as defined in [B-B], then $\text{rad } N$ is a nilideal. If, additionally, R does not contain any non-zero nilpotent elements, then $\text{rad } N$ is the maximal nilideal of A , that means the radical [B-B, Lemma 5].

For quadratic forms permitting composition, the radical can be annihilated by a suitable exponent which depends on the dimension of A ([K-S] or [M2]). For degenerate cubic forms permitting composition of the kind $N(a, x) = aN_C(x)$ where N_C is a quadratic form permitting composition, the radical can be annihilated by exactly that exponent which depends on the dimension of C , since in that case $\text{rad } N_0 = 0 \oplus \text{rad } N_C$.

8.2. Let $X = \mathbb{P}_R^n$ be the n -dimensional projective space over R , that is $X = \text{Proj } S$ where $S = R[t_0, \dots, t_n]$ is the polynomial ring in $n+1$ variables over R , equipped with the canonical grading $S = \bigoplus_{m \geq 0} S_m$. We have $\text{rank } S_m = \binom{m+n}{n}$. We know that $\mathcal{O}_X(m)$ is a locally free \mathcal{O}_X -module of rank one for each $m \in \mathbb{Z}$ and

$$H^0(X, \mathcal{O}_X(m)) = S_m \text{ for } m \geq 0,$$

$$H^0(X, \mathcal{O}_X(m)) = 0 \text{ for } m < 0.$$

Example 4. Let \mathcal{C} be the split octonion algebra

$$\text{Zor}(\mathcal{O}_X(l) \oplus \mathcal{O}_X(m) \oplus \mathcal{O}_X(-l-m), \alpha)$$

over X with norm $n_{\mathcal{C}}$ as defined in [P, 3.3] (l, m positive integers). Let $\mathcal{A} = \mathcal{O}_X \oplus \mathcal{C}$ and $N((x_1, x_2)) = x_1 n_{\mathcal{C}}(x_2)$ for all sections x_1 in \mathcal{O}_X , x_2 in \mathcal{C} . Then $N(1) = 1$, N permits composition, and N is absolutely indecomposable. We get

$$A = H^0(X, \mathcal{A}) = R \oplus H^0(X, \mathcal{C}) = R \oplus \begin{bmatrix} R & S_l \oplus S_m \\ S_{l+m} & R \end{bmatrix}$$

with the algebra multiplication in $H^0(X, \mathcal{C})$ as described in [P, 3.8]. A is an alternative R -subalgebra of $S \oplus \text{Zor } S$ of rank

$$3 + \binom{l+n}{n} + \binom{m+n}{n} + \binom{(l+m)+n}{n}$$

and $x^3 - T_1(x)x^2 + T_2(x)x - T_3(x)1 = 0$ for each $x \in A$. If $n = 1$ then $\text{rank}_R H^0(X, \mathcal{A}) = 6 + 2(l+m) \geq 10$ must be even. If R is a field then the cubic form $N_0 = N(X)$ restricted to the subalgebra

$$R \oplus \begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix}$$

of rank 3 is nondegenerate and

$$\text{rad } N_0 = 0 \oplus \text{rad}(H^0(X, \mathcal{C})) = 0 \oplus \begin{bmatrix} 0 & S_l \oplus S_m \\ S_{l+m} & 0 \end{bmatrix}$$

is the radical of A [P, 3.8]. We have

$$(\text{rad } N_0)^2 = 0 \oplus \begin{bmatrix} 0 & 0 \\ S_{l+m} & 0 \end{bmatrix}$$

and $(\text{rad } N_0)^3 = 0$.

Example 5. Let $\mathcal{F} = \mathcal{O}_X(m_1) \oplus \mathcal{O}_X(m_2) \oplus \mathcal{O}_X(m_3)$, then $\mathcal{A} = \mathcal{E}nd_X(\mathcal{F})$ is an Azumaya algebra over X of constant rank 9. We have

$$\mathcal{A} = \begin{bmatrix} \mathcal{O}_X & \mathcal{O}_X(a) & \mathcal{O}_X(b) \\ \mathcal{O}_X(-a) & \mathcal{O}_X & \mathcal{O}_X(b-a) \\ \mathcal{O}_X(-b) & \mathcal{O}_X(a-b) & \mathcal{O}_X \end{bmatrix}$$

with $a = m_1 - m_2$, $b = m_1 - m_3$, the right hand side being equipped with the usual matrix multiplication. $H^0(X, \mathcal{A})$ is a unital associative R -algebra of degree 3 which admits a cubic form $N_0 = N(X) : H^0(X, \mathcal{A}) \rightarrow H^0(X, \mathcal{O}_X)$ permitting composition which satisfies $N_0(1_A) = 1$.

(1) If $a, b > 0$ and $b - a > 0$ then

$$H^0(X, \mathcal{A}) = \begin{bmatrix} R & S_a & S_b \\ 0 & R & S_{b-a} \\ 0 & 0 & R \end{bmatrix}$$

has rank

$$3 + \binom{a+n}{n} + \binom{b+n}{n} + \binom{(b-a)+n}{n}.$$

In particular, if $n = 1$ then $\text{rank}_R H^0(X, \mathcal{A}) = 6 + 2b \geq 8$ must be even. If R is a field then

$$\text{rad } N_0 = \begin{bmatrix} 0 & S_a & S_b \\ 0 & 0 & S_{b-a} \\ 0 & 0 & 0 \end{bmatrix}$$

is the radical of N_0 , i.e. the radical of A . We get

$$(\text{rad } N_0)^2 = \begin{bmatrix} 0 & 0 & S_b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and $(\text{rad } N_0)^3 = 0$.

(2) If $a = b > 0$ then

$$H^0(X, \mathcal{A}) = \begin{bmatrix} R & S_a & S_a \\ 0 & R & R \\ 0 & R & R \end{bmatrix}$$

has odd rank $5 + 2\binom{a+n}{n}$. In particular, if $n = 1$ then $\text{rank}_R H^0(X, \mathcal{A}) = 5 + 2(a+1) \geq 9$. If R is a field then

$$\text{rad } N_0 = \begin{bmatrix} 0 & S_a & S_a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and $(\text{rad } N_0)^2 = 0$. Again, $\text{rad } N_0$ is the radical of A .

(3) If $a = 0$ and $b > 0$ then

$$H^0(X, \mathcal{A}) = \begin{bmatrix} R & R & S_b \\ R & R & S_b \\ 0 & 0 & R \end{bmatrix}$$

has odd rank $5 + 2\binom{b+n}{n}$. In particular, if $n = 1$ then $\text{rank}_R H^0(X, \mathcal{A}) = 5 + 2(b+1) \geq 9$. If R is a field then

$$\text{rad } N_0 = \begin{bmatrix} 0 & 0 & S_b \\ 0 & 0 & S_b \\ 0 & 0 & 0 \end{bmatrix}$$

and $(\text{rad } N_0)^2 = 0$.

Analogously, one can construct classes of degenerate forms of degree higher than 3 over R which permit composition.

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